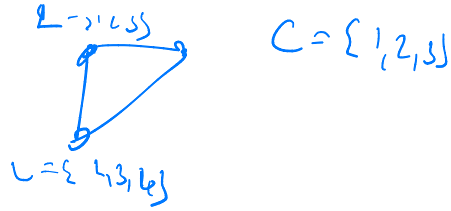
Lecture 8: List Coloring

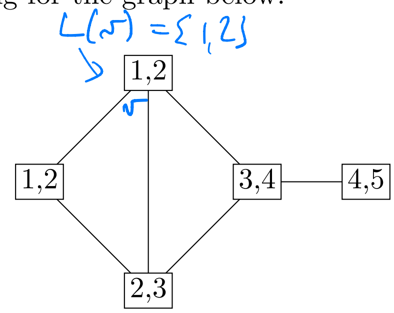
Introduced by Vizing (1976) and independently by Erdős, Rubin, and Taylor (1979).

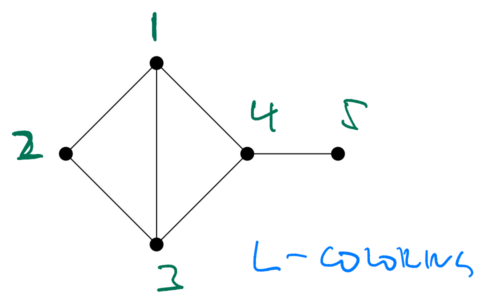
A *list assignment* of G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of colors. The elements of the list $L(v)$ are called *admissible colors* for the vertex v. An *L-coloring* is a mapping $\varphi : V(G) \to \bigcup_v L(v)$ such that



- $\varphi(v) \in L(v)$ for every $v \in V(G)$, and
- $\varphi(u) \neq \varphi(v)$ whenever u and v are adjacent vertices of G.

1: Find a list coloring for the graph below.

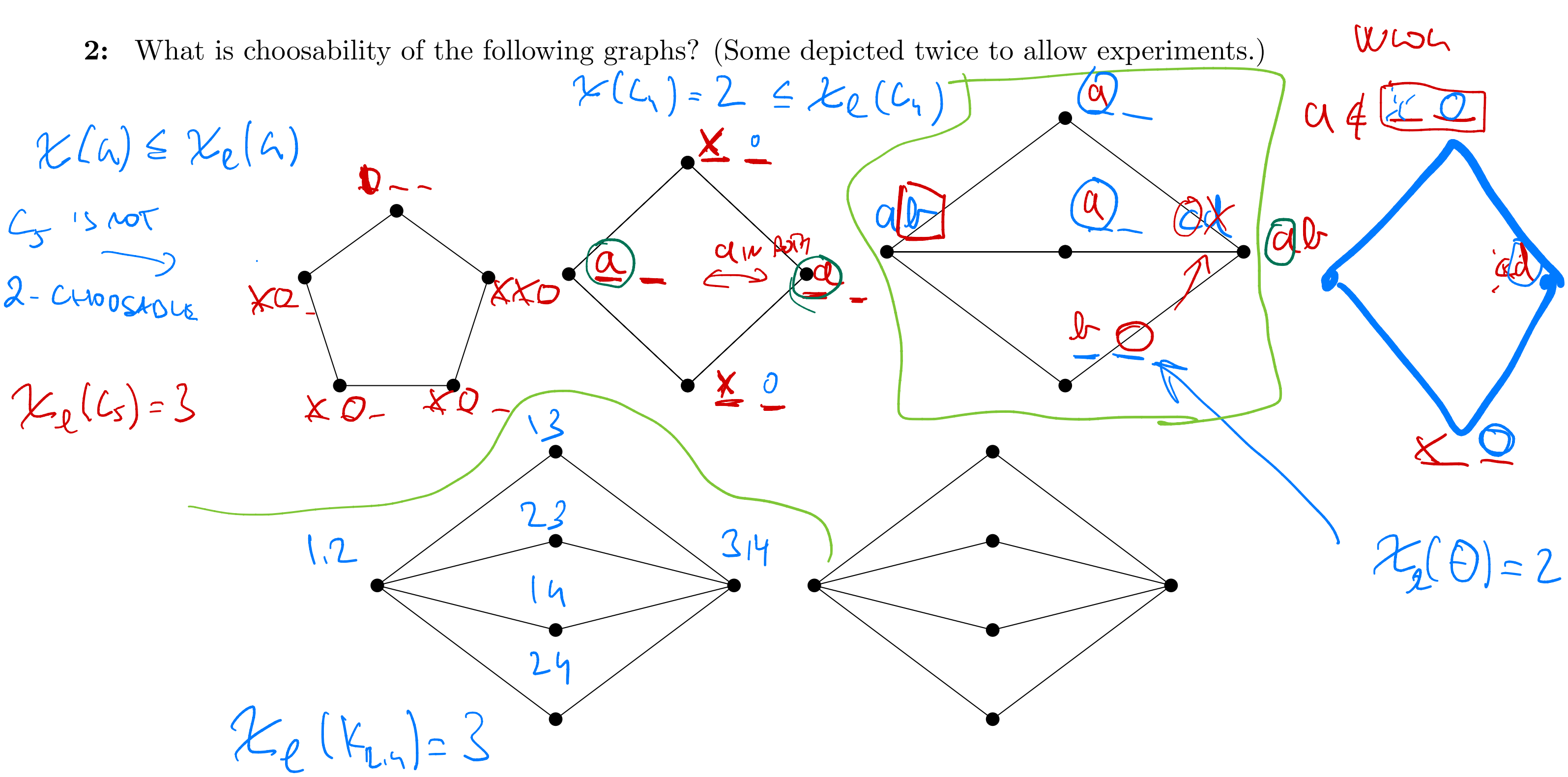




For a graph G we say

- *L-colorable* if G admits an L-coloring
- *k-choosable* if, for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$, G is L-colorable.
- *choosability* of a graph G, denoted by $\mathrm{ch}(G)$ or $\chi_\ell(G)$, is the smallest k such that G is k-choosable.

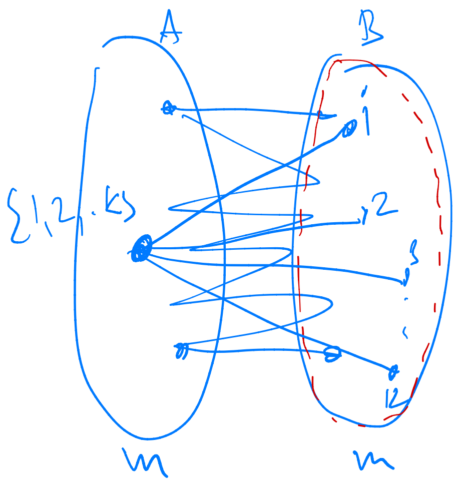
2: What is choosability of the following graphs? (Some depicted twice to allow experiments.)



Erdős, Rubin and Taylor showed that there are bipartite graphs with arbitrary large list chromatic number.

Theorem 1 (Erdős, Rubin and Taylor). *For $m \geq \binom{2k-1}{k}$, the bipartite graph $K_{m,m}$ is not k-choosable.*

3: Prove the theorem. Hint: Use the last graph from previous exercise.

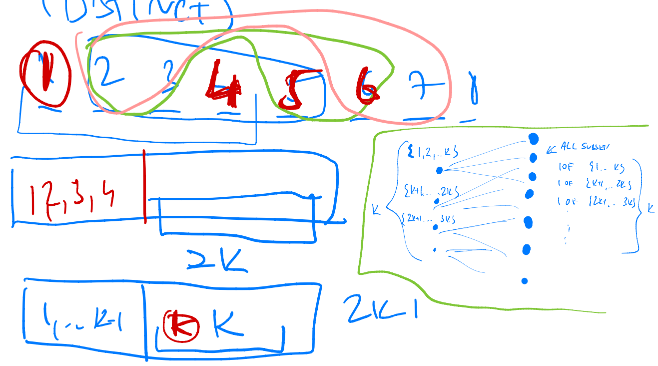


$(1,2,\ldots 2k-1)$

$L(v) \in \binom{[2k-1]}{k}$

AND USE ALL SUBSETS IN EACH A & B

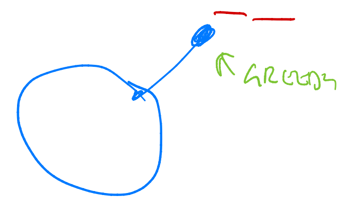
IN L-COLORING AT LEAST K COLORS USED ON B (DISTINCT)



Let G be a graph from which we start removing vertices of degree one consecutively one by one. By this procedure, we ends-up either by a vertex of minimum degree ≥ 2 or by a single vertex. We denote the resulting graph by core(G). Recall that the Theta graph $\Theta_{a,b,c}$ is comprised from two vertices that are connected by three paths of length a, b, and c that are pair-wise disjoint except at the end-vertices.

Theorem 2 (Rubin). *A graph G is 2-choosable if and only if*

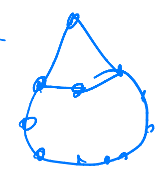
$$\text{core}(G) \in \{K_1, C_{2m+2}, \Theta_{2,2,2m} : m \geq 1\}.$$



4: Show that

$$\text{core}(G) \in \{K_1, C_{2m+2}, \Theta_{2,2,2m} : m \geq 1\}$$

are 2-choosable.



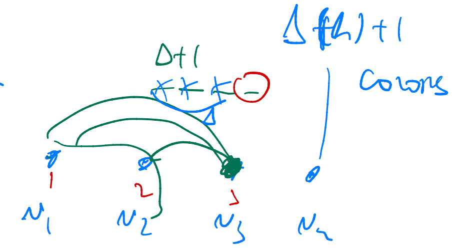
1 List version of Brooks theorem and beyond

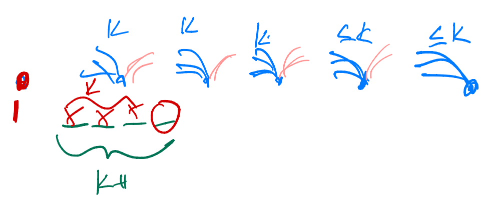
5: Show that

$$\chi_\ell(G) \leq \Delta(G) + 1.$$

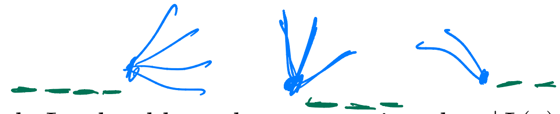
Moreover, k-degenerated graph is $(k+1)$-choosable.

GREEDY COLORING WORK





6: State Brook's theorem. $\chi(G) = \Delta(G)+1$ IFF G IS K_n C_{2n+1} FOR $n \geq 1$



Question: When is graph L-colorable under assumption that $|L(v)| \geq d(v)$ for every vertex v? We call such an assignment a *degree* list assignment.

Lemma 3. *Let (G, L) be a pair of a connected graph and L a degree list assignment such that G is not L-colorable. Then following hold:*

(1). *$|L(v)| = d(v)$ for every vertex v of G;*

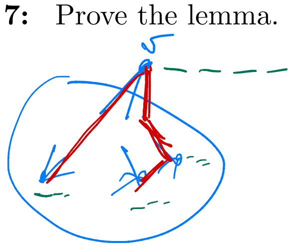
(2). *If u and v are two adjacent vertices of G and u is not a cut-vertex then $L(u) \subseteq L(v)$;*

(3). *If G is 2-connected then it is an odd cycle or a complete graph and L assigns the same $\Delta(G)$ colors to all vertices.*

7: Prove the lemma.

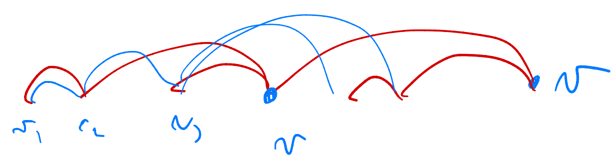
SPANNING TREE

①



I WANT G L-COLORABLE

$(|L(v)| > d(v))$



GREEDILY

② OR

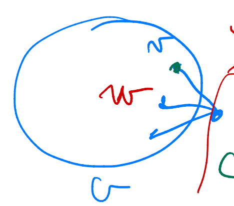
u NOT CUTVERTEX

$L(u) \not\subseteq L(v)$

THEN G IS L-COLORABLE

$\Rightarrow d(u) \leq d(v)$

$\leq (d-1)$ $|L(u)| = d$



G−u

& REMOVE C(u) FROM L(w) IF uw ∈ E(G)

$C(u) \notin L(v)$

w IN G−u $\therefore d_{G-u}(w) = d_G(w) - 1$

$\therefore |L_{G-u}(w)| \geq |L(w)| - 1$

BUT $|L_{G-u}(v)| = |L(v)|$

$> d_{G-u}(v)$

③ G IS d-REGULAR

& ALL LISTS ARE THE SAME

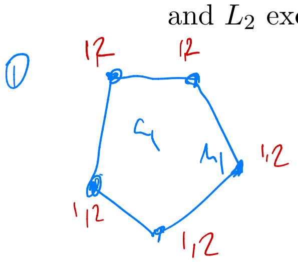
WHEN NOT Δ-COLORABLE

⇒ BROOKS ⇒ C_{2n+1} OR K_n

We define a class of pairs $(G, L) \in \mathcal{G}_{\mathcal{L}}$ whenever

1. G is an odd cycle and L assigns a same two colors to every vertex of G; or
2. G is a complete graph and L assigns a same colors to every vertex of G; or
3. there exists pairs (G_1, L_1) and (G_2, L_2) from $\mathcal{G}_{\mathcal{L}}$ such that G is obtained from identifying a vertex u_1 from G_1 with a vertex u_2 from G_2 into a vertex u of G, where $L_1(u_1) \cap L(u_2) = \emptyset$. And, L coincides with L_1 and L_2 except at the identified vertex u it has list $L(u) = L_1(u_1) \cup L(u_2)$.

①



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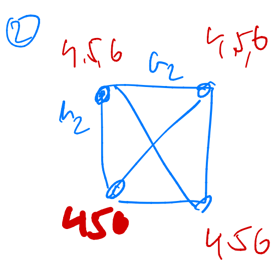
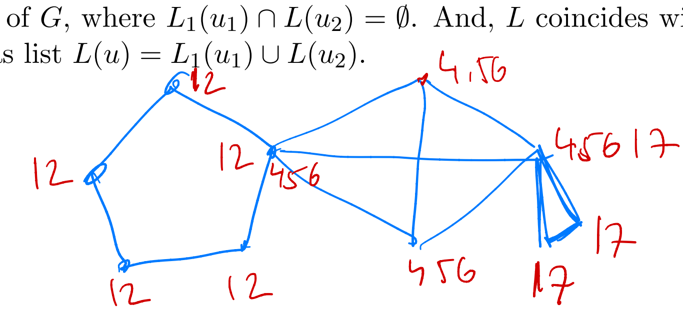
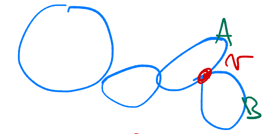




Theorem 4. *A connected graph G is not L-colorable for a degree list assignment L if and only if $(G, L) \in \mathcal{G}_{\mathcal{L}}$.*

8: Hint: Prove the theorem.

IF $(G,L) \in \mathcal{G}_L$ THEN G NOT L-COLORABLE



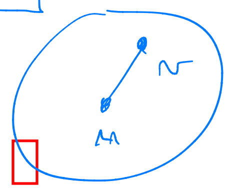
BY IF OF 2-CONNECTED COMPONENTS SUPPOSE $\exists\, \varphi$ L-COLORING

$\varphi(v) \in A$

NOT L-COLORABLE $\Rightarrow$ GALLAI TREE.

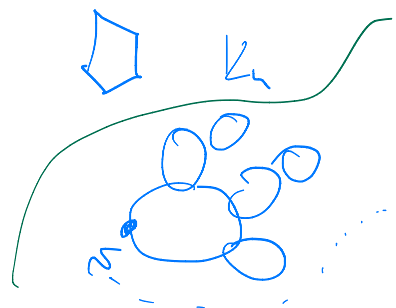
$\boxed{\text{LEMMA 3}}$

$\varphi(v) \in B$

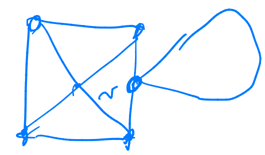


G 2-CONNECTED

LEMMA 3-3



$\boxed{L(v)}$

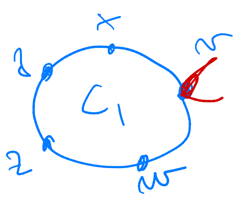


$\to G - v$

CUT VERTICES

$c \in L(v)$

$\varphi(v) = c$



$L(x) \leq L(v)$

$L(y) = L(x) \Leftrightarrow d(x) = d(y).$

$\left.\begin{array}{l} L(x) \leq L(y) \\ L(y) \leq L(x) \end{array}\right\} L\,3.2$

$\forall\, x,y \in C_1 \setminus v$

$L(y) = L(\cancel{x}) \in L(v)$

$d(x) = d(y)?$

$c(y) = 2$

This asserts our first proposition.

Corollary 5. *For any graph G that is not an odd cycle or a completele graph, holds*

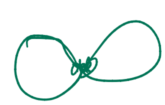
$$\chi_\ell(G) \leq \Delta(G).$$

9: Prove the corollary.

LET G HAVE LIST ASSIGNMENT L $|L(v)| = \Delta(G)$ $\forall v$

& G NOT L-COLORABLE

IF $\forall v$ $|L(v)| \geq d(v)$ L. 3 APPLIES



$\to$ IF 2-CONNECTED $\Rightarrow$ ODD CYCLE OR COMPLETE GRAPH

L 3.1 G IS Δ-REGULAR & GALLAI TREE IS MUST BE 2-CONNECTED

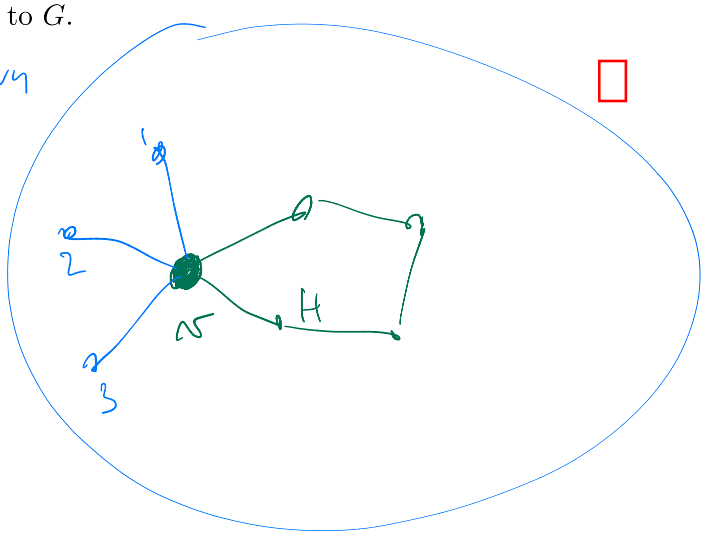
Theorem 6 (Gallai)**.** *In a k-critical graph with $k \geq 4$, low vertices induce a forest (possibly empty) whose blocks are odd cycles and complete graphs.*

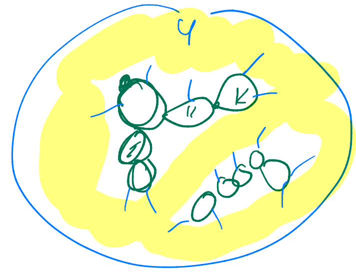
K-CRITICAL GRAPH

Recall that low vertices are vertices of degree $k-1$.

10: Prove the theorem. Hint: Consider G without low vertices L and try to extend $(k-1)$ coloring of $G-L$ to G.

TRYING TO K-1 COLOR





K-CRITICAL GRAPH G

$\delta(G) \geq \boxed{k-1}$

$d(v) = k-1$

"$L(v) = k-1$"

$L(v) = [1, \ldots, k-1] \setminus \{\varphi(x),\ x \in N(v) \cap V(G-L)\}$

$\Rightarrow L(v) \geq d_H(v)$ BUT L CANNOT COLOR H $\Rightarrow$ H IS A GALLAI TREE

$d(v) = k+2$ "$L(v) = k-1$" $\rightarrow$ $d(v) = k+1$, $L(v) = k-2$ $\quad d(v) \cancel{\leq} |L(v)|$

1.1 Planar graphs

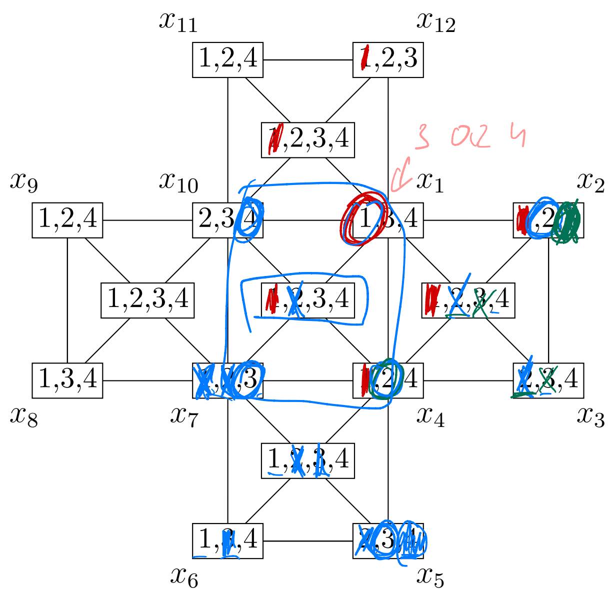
Theorem 7 (Thomassen)**.** *Every planar graph is 5-chosable.*

The next lemma implies the above theorem. Recall that a near-triangulation is a plane graph whose all inner faces are 3-cycles.

Lemma 8. *Let G be a 2-connected near-triangulation and let $C = x_1x_2\cdots x_nx_1$ be the outerface. Let L be a list-assignment of G such that $|L(x)| \geq 3$, for $x \in V(C)$, and otherwise $|L(x)| \geq 5$. Suppose that c is an L-coloring of x_1 and x_n. Then, c can be extended to an L-coloring of G.*

11: Prove the lemma by induction.

Voigt construct a non-4-choosable planar graph on 238 vertices. Later Mirzakhani (the famous one) such a graph on 63 vertices. A gadget of her construction is depicted below.



12: Show that the graph above is not list-colorable and the graph on the next page is also not list-colorable.

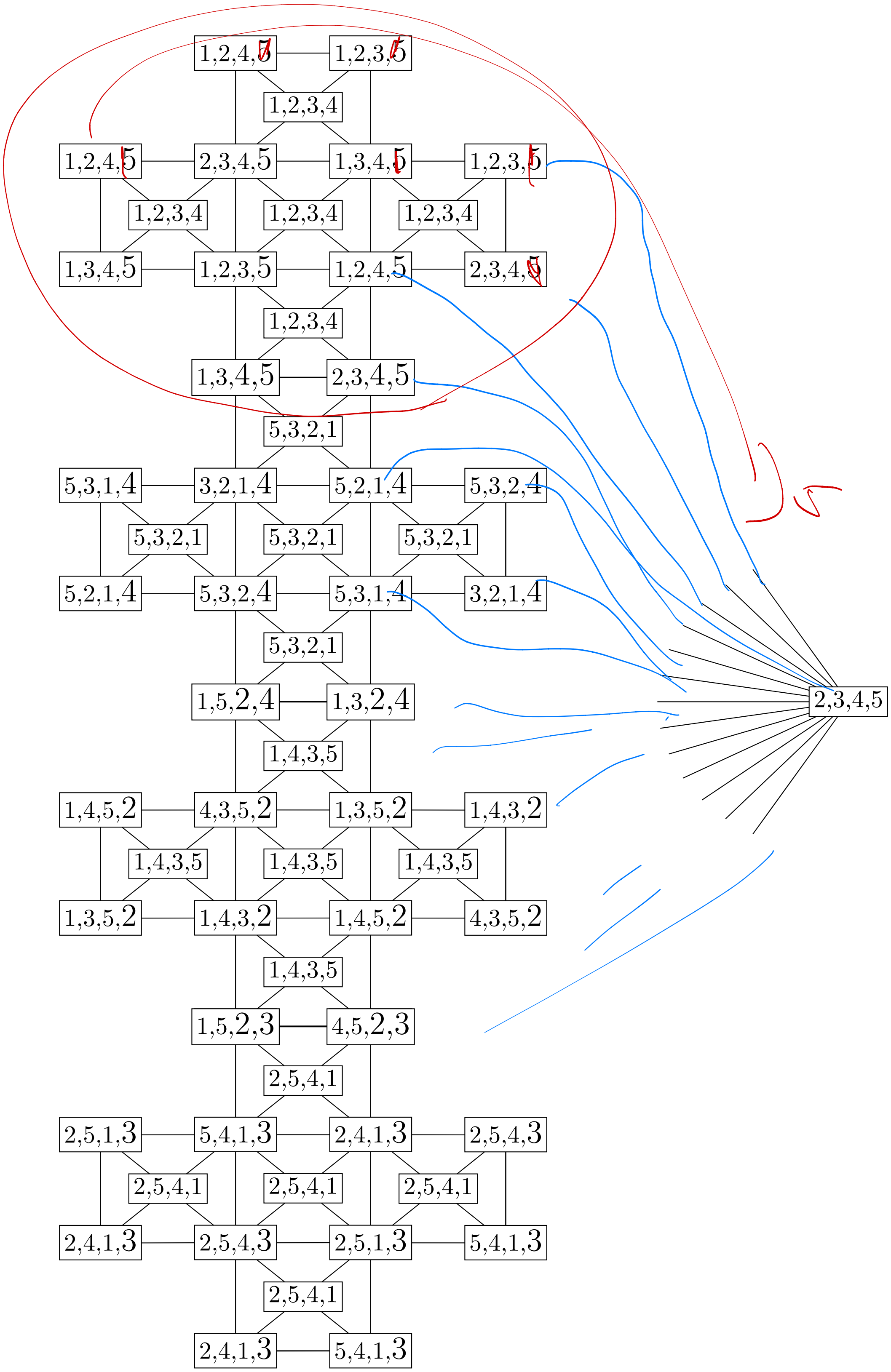


Figure 1: Mirzakhani construction.